NON-COMMUTATIVE SPHERES III: IRRATIONAL ROTATIONS

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Dedicated to Professor Huzihiro Araki on the occasion of his 60'th birthday.

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Abstract

Let A_θ be the irrational rotation algebra i.e. the C^*-algebra generated by two unitaries U, V satisfying $VU = e^{2\pi i\theta}UV$, with θ irrational, and consider the fixed point subalgebra B_θ under the flip automorphism $U \to U^{-1}$, $V \to V^{-1}$. We prove that B_θ is an AF-algebra.

CONTENTS

Chapter 1

Introduction

In this paper we continue the study, begun in [BEEK 1] and [BEEK 2], of the fixed point subalgebra of the rotation algebra under the flip. Recall from [Rie] that the rotation algebra A_θ is the universal C*-algebra generated by two unitaries U, V satisfying $VU = \rho UV$, where $\rho = e^{2\pi i\theta}$ and $0 \leq \theta < 1$. The flip σ is the automorphism of this algebra defined through the requirements

$$\sigma(U) = U^{-1}, \;\; \sigma(V) = V^{-1}. \tag{1.1}$$

enote the fixed point algebra under the flip by B_θ, and the crossed product by C_θ. In [BEEK 1] it was established that if θ is irrational, then B_θ is the universal C*-algebera generated by two self-adjoint elements a, b satisfying

$$ba^2 + a^2b = 2\lambda aba + 4(1-\lambda^2)b \tag{1.2}$$

$$ab^2 + b^2a = 2\lambda bab + 4(1-\lambda^2)a \tag{1.3}$$

$$baba = (4\lambda^2 - 1)abab - 2\lambda a^2b^2 + 8\lambda(1-\lambda^2)(a^2 + b^2 - 1) \tag{1.4}$$

where $\lambda = \cos(2\pi\theta)$. This result was extended to rational $\theta \notin \{0, \frac{1}{2}\}$ in [BEEK 2] while the universal C*-algebra fails to exist if $\theta \in \{0, \frac{1}{2}\}$. The connection between a, b and U, V is

$$a = U + U^{-1}, \;\; b = V + V^{-1}. \tag{1.5}$$

When $\theta = p/q$ is rational, it was proved in [BEEK 2] that B_θ is the subalgebra of the C*-algebra $C(S^2, M_q)$ of continuous functions from the 2-sphere S^2 into the algebra of complex $q \times q$ matrices M_q determined up to isomorphism as follows: There are four distinct points ω_o, ω_1, ω_2 and ω_3 in S^2 and to each point ω_i is associated a self-adjoint projection P_i in

M_q. The dimensions of P_i are all $\frac{q-1}{2}$ when q is odd, and when q is even, $\dim(P_0) = \frac{q-2}{2}$ whilst $\dim(P_i) = \frac{q}{2}$ for $i = 1,2,3$. The algebra B_θ consists of those functions $f \in C(S^2, M_q)$ such that $f(\omega_i)$ commutes with P_i for $i = 0,1,2,3$.

An analogous result was proved for C_θ, with the difference that M_q is replaced by M_{2q}, and $\dim P_i = q$ for $i = 0,1,2,3$, independently of the parity of q. (These latter results were extended to other finite subgroups of the canonical action of $SL(2, Z)$ on A_θ by Farsi and Watling, [FW1], [FW2], [FW3], [FW4].)

When θ is irrational, the algebras B_θ and C_θ are simple with a unique trace state, [BEEK 1]. Furthermore,

$$K_0(C_\theta) \cong Z^6$$
$$K_1(C_\theta) \cong 0$$

for all θ, [Kum 2]. A direct argument when θ is rational is given in [BEEK 2]. In this paper we will prove

Theorem 1.1 *The algebras B_θ and C_θ are AF-algebras when θ is irrational.*

Since B_θ is a corner of C_θ, it suffices to show this for C_θ. In [BEEK 2] we expressed some hope of proving this by approximation by rational θ, but as it is we do not do this directly, but rather use Putnam's tower construction [Put] very much as in [BEK], together with a method of constructing projections in C_θ which was devised by Kumjian, [Kum 1], modifying Rieffel's method of constructing projections in [Rie].

On the way to proving Theorem 1.1 we will show that C_θ is an inductive limit of finite direct sums of certain subhomogeneous algebras over the unit interval and some full matrix algebras; see Corollary 7.4 and (7.1)–(7.5). That C_θ is AF will follow from this by combining with techniques from [BBEK] and [Su]. The strategy is to use unique trace state and simplicity to prove small eigenvalue variation for the inductive limit.

We can also classify the C_θ's, essentially as the A_θ's, by computing the range of the trace:

Theorem 1.2 *If $0 < \theta_1,\ \theta_2 < 1$ and $\theta_1,\ \theta_2$ are irrational, then C_{θ_1} is isomorphic to C_{θ_2} if and only if $\theta_1 \in \{\theta_2, 1-\theta_2\}$.*

This contrasts with the rational case where the algebras $C_{p/q}$ and $C_{p'/q'}$ (with p,q, and also p',q', relatively prime) are isomorphic if and only if $q = q'$, [BEEK 2].

The proof of Theorem 1.2 is independent of the rest of this paper, and is as follows: Since any projection in B_θ is a projection in A_θ, and the Rieffel projection in A_θ has a representative which is flip invariant, it follows that the range of the trace on the projections in B_θ is the same as in A_θ, which is $(Z + Z\theta) \cap [0, 1]$. But B_θ is isomorphic to $eC_\theta e$, where e is a projection in C_θ with trace $1/2$, and hence the range of the trace on C_θ is $\frac{1}{2}(Z + Z\theta) \cap [0, 1]$. Thus, if C_{θ_1} and C_{θ_2} have the same range of the trace, then $\theta_1 = \theta_2$ or $\theta_1 = 1 - \theta_2$, and hence C_{θ_1} and C_{θ_2} are non-isomorphic unless θ_1 and θ_2 are related in this way. On the other hand C_θ and $C_{1-\theta}$ are isomorphic since the isomorphism $u \to v$, $v \to u$ of A_θ and $A_{1-\theta}$ intertwines the flips of those two algebras. This proves Theorem 1.2.

Chapter 2

Putnam's tower construction on T

In this section we will use the identification $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$, and by the term interval in $\boldsymbol{T}$ we will mean closed nonempty intervals where both endpoints (which are supposed to be distinct) lie in the orbit $\boldsymbol{Z}\,\theta$ mod 1, where $0 < \theta < 1$ is a fixed irrational number. By a partition of $\boldsymbol{T}$ will be meant a finite collection of closed intervals with union $\boldsymbol{T}$ such that the intersection of any pair of the intervals consists of at most one point (which is then an endpoint of both the intervals and thus is contained in $\boldsymbol{Z}\theta$). Note that the set of intervals are left globally invariant under both α and σ, where

$$\alpha(t) = t + \theta, \tag{2.1}$$

and

$$\sigma(t) = -t. \tag{2.2}$$

In particular we will consider the partitions of $\boldsymbol{T}$ determined by the requirements that

$$-(N-1)\theta, -(N-2)\theta, ..., -\theta, 0, \theta, ..., (N-2)\theta, (N-1)\theta$$

shall be the set of endpoints, where N is a positive integer. In particular, we will see that these partitions arise from a Putnam tower construction with 3 towers (unless N is very small). For later use, we will choose N in a specific way:

For any positive integer M, choose $\delta > 0$ so small that all the translates of $\langle \theta/2 - \delta, \theta/2 + \delta \rangle$ by $m\theta$, with $|m| \leq M + 1$, are pairwise disjoint on $\boldsymbol{T}$. Then choose $N > 0$ such that

$$N\theta \in \langle \theta/2, \theta/2 + \delta \rangle \tag{2.3}$$

and such that the orbit piece $\{-(N-1)\theta, ..., (N-1)\theta\}$ intersects both the intervals $\langle \theta/2, N\theta \rangle$ and $\langle N\theta, \theta/2 + \delta \rangle$. Now let $k\theta$ denote the point in the orbit piece in $\langle N\theta, \theta/2 + \delta \rangle$ which is closest to $N\theta$, and $l\theta$ the point in the orbit piece in $\langle\theta/2, N\theta\rangle$ which is closest to $N\theta$. Thus, $|k| < N$, $|l| < N$ and $[N\theta,\ k\theta]$, $[l\theta,\ N\theta]$ are elements in the partition of $\boldsymbol{T}$ determined by $\{-N\theta, ..., N\theta\}$, while $[l\theta,\ k\theta]$ is an element in the partition of $\boldsymbol{T}$ determined by $\{-(N-1)\theta,\ (N-1)\theta\}$.

Lemma 2.1 *Define M, δ, N, l, k as above. Then*

$$k + l < 0 \tag{2.4}$$

and the partition of $\boldsymbol{T}$ defined by the orbit piece $\{-(N-1)\theta, ..., (N-1)\theta\}$ consists of the intervals

$$\begin{aligned} [l\theta, k\theta] + m\theta &\quad , \quad 0 \le m \le -(k+l), \\ [(-k+1)\theta, (-N+1)\theta] + m\theta &\quad , \quad 0 \le m \le N+k-2, \\ [(-N+1)\theta, (-l+1)\theta] + m\theta &\quad , \quad 0 \le m \le N+l-2. \end{aligned} \tag{2.5}$$

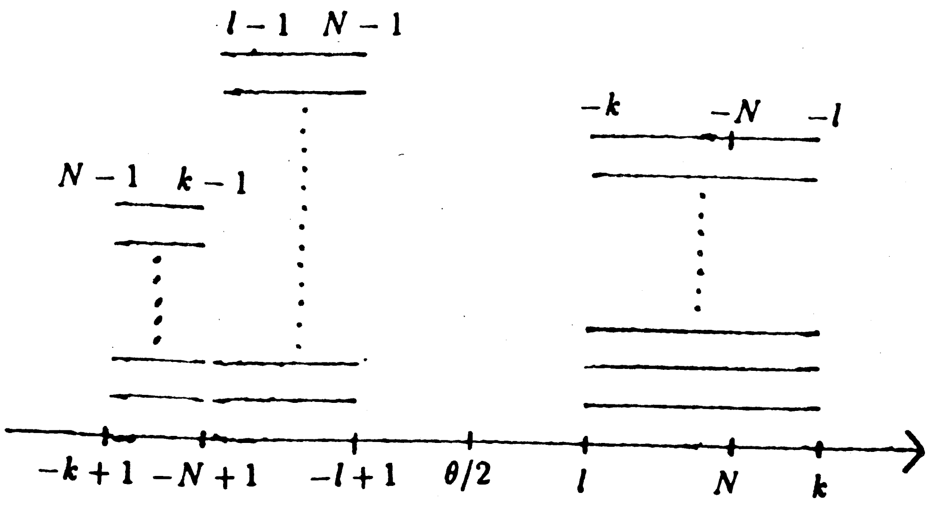
Furthermore, this partition consists of the Putnam towers associated to the $\alpha\sigma$-invariant set

$$[(-k+1)\theta,\ (-l+1)\theta] \cup [l\theta,\ k\theta]. \tag{2.6}$$

This set is contained in the interval $\langle\theta/2-\delta,\ \theta/2+\delta\rangle$, and the heights of the three towers are all at least $2M+2$.

Proof

For clarity, let us draw a figure of the whole tower construction (drawn in the case that $k < l$):



(2.7)

Here, any integer label n refers to the point $n\theta$. Inspection of the figure above shows that the set of left end points of the intervals occurring runs through the set $\{m;\ -N+1 \leq m \leq N-1\}$ and each number of this set occurs exactly once. The same is true for the set of right endpoints. Hence all we have to show is that the interiors of the floors of the towers indicated above do not overlap, that is, if $n\theta$ lies in the interior of some floor, then $|n| \geq N$. We check this for the three towers separately.

Tower 1 from the right: As for the basement, note that the only $n\theta$ in $\langle l\theta,\, k\theta\rangle$ with $|n| \leq N$ is $N\theta$, by the definition of k and l. For the remaining floors $\langle l\theta,\, k\theta\rangle + m\theta$ we proceed by induction with respect to m. If

$$n\theta \in \langle l\theta,\, k\theta\rangle + m\theta \tag{2.8}$$

with $|n| \leq N-1$ and $m \geq 1$, then

$$(n-1)\theta \in \langle \theta, k\theta\rangle + (m-1)\theta, \tag{2.9}$$

and hence, by the induction hypothesis, we must have $n = -N+1$ and $m-1 > 0$. But as σ, applied to $l\theta, N\theta$ and $k\theta$, gives $-l\theta$, $-N\theta$, $-k\theta$ respectively, and the whole set $\{-N\theta,\, (-N+1)\theta, ..., (N-1)\theta, N\theta\}$ is σ-invariant, it follows that

$$\langle l\theta,\, k\theta\rangle \,+\, (m-1)\theta = \langle -k\theta, -l\theta\rangle \tag{2.10}$$

for this m, whence

$$m-1 = -(k+l).$$

This proves simultaneously that

$$k+l < 0$$

and that the statement for the first tower holds.

Towers 2 and 3: Note that α maps the roof of Tower 1 onto the union of the basements of Towers 2 and 3, and that hence the only point of the form $n\theta$ in $\langle(-k+1)\theta, (-l+1)\theta\rangle$ with $|n| \leq N-1$ is $(-N+1)\theta$. This is seen by subtracting θ and using that $-N\theta$ is the only point of the form $n\theta$ with $|n| \leq N$ in $< -k\theta,\, -l\theta >$. For the remaining floors of e.g. Tower 2, i.e., $\langle(-N+1)\theta, (-l+1)\theta\rangle + m\theta$, we proceed by induction again: If

$$n\theta \in \langle(-N+1)\theta,\, (-l+1)\theta\rangle + m\theta,$$

then

$$(n-1)\theta \in \langle(-N+1)\theta,\ (-l+1)\theta\rangle + (m-1)\theta$$

and hence $n = -N+1$ by the induction hypothesis. Thus,

$$-N\theta \in \langle(-N+1)\theta,\ (-l+1)\theta\rangle + (m-1)\theta$$

for this m. Since the neighbouring points of $-N\theta$ in $\{-N\theta, ..., N\theta\}$ are $-k\theta$ and $-l\theta$ it follows that

$$(-N+1)+(m-1) = -k \text{ and } (-l+1)+(m-1) = -l,$$

from which follows

$$m = N-k \qquad \text{and} \qquad m = 0,$$

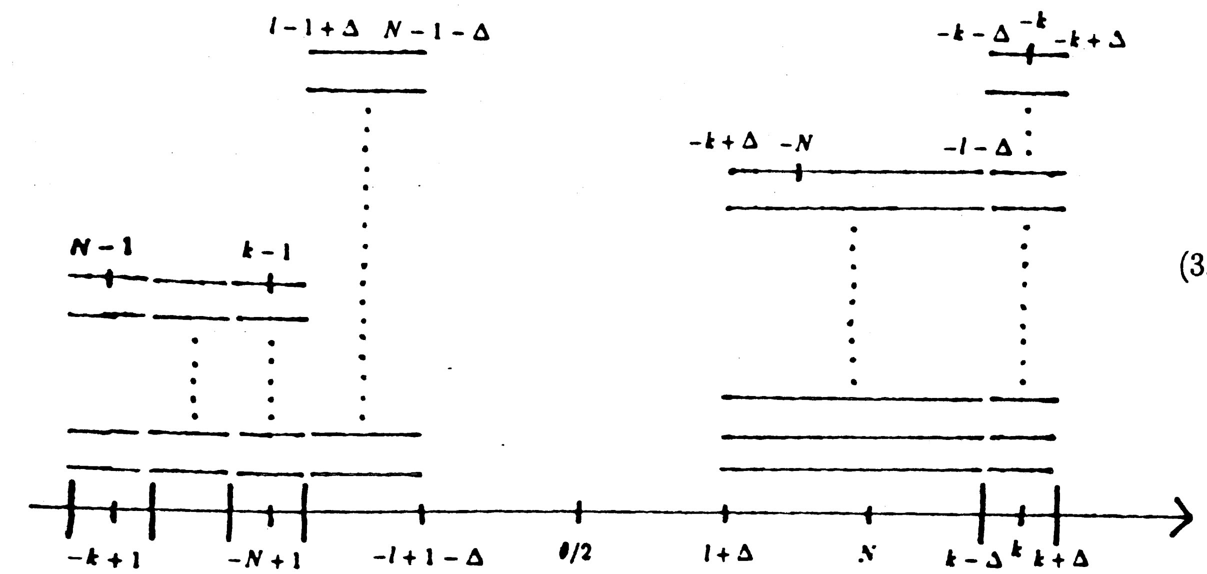
which is a contradiction. Thus the only restriction on the range of m is that $(-N+1)+m$ and $(-l+1)+m$ should lie in $\{-N+1, ..., N-1\}$, i.e. $(-l+1)+m \leq N-1$, i.e. $m \leq N+l-2$. Tower 3 is treated analogously.

Finally, since δ was chosen such that all translates of $\langle\theta/2-\delta,\ \theta/2+\delta\rangle$ by $m\theta$, with $|m| \leq M+1$, are pairwise disjoint, and all three basements are contained in this set, it follows that any translate of any basement by $m\theta$, with $|m| \leq M+1$, cannot intersect any other basement. It follows that the height of each of the three towers is at least $2M+2$.

Chapter 3

A subsidiary tower construction

In order to construct finite-dimensional subalgebras of $C_\theta = C(T) \times_\alpha Z \times_\sigma Z_2$, we will have to modify the three-tower construction in Lemma 2.1 and replace it by a six-tower construction. In the case that $k < l$, the new tower construction looks as follows:



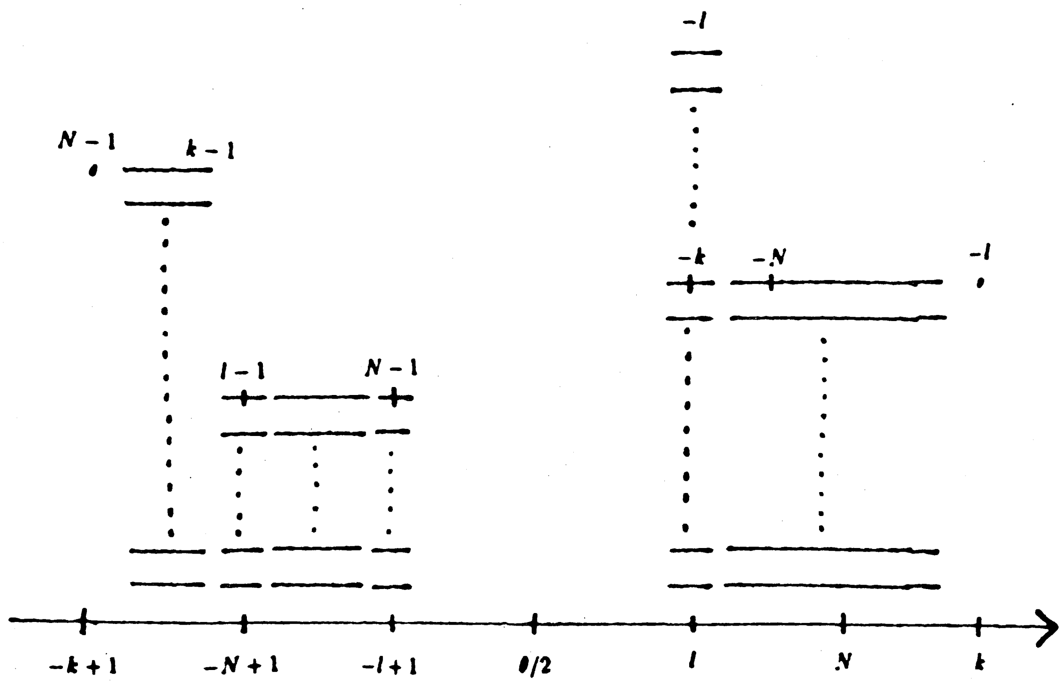
(3.1)

Here, Δ is a nonzero integer such that $\Delta\theta$ is much closer to 0 in T than any of the points in the orbit $\{-(N-1)\theta, ..., (N-1)\theta\}$ are to each other. For definiteness, let us assume that (mod 1)

$$0 < \Delta\theta \leq \frac{1}{4}\min\{(-l+1)\theta - (-N+1)\theta,\ (-N+1)\theta - (-k+1)\theta\}. \qquad (3.2)$$

It is then easily verified that the depicted tower construction really is a Putnam tower construction over the basement $[(-k+1-\Delta)\theta, (-l+1-\Delta)\theta] \cup [(l+\Delta)\theta, (k+\Delta)\theta]$. This basement is still $\alpha\sigma$-invariant ($\alpha\sigma$interchanges the two pieces). Note also that σ maps each of the six towers into themselves except for the first and third tower from the left, which are interchanged, and σ reverses the order of the floors, in particular interchanging basements and roofs.

In the case that $l < k$, we use the following new tower construction:



(3.3)

The same remarks, with the obvious modifications, apply to this construction.

In any case, let Y_1, Y_2, Y_3 denote the three ground floors of the wide towers, i.e., towers number 2, 4 and 5 from the left in figure (3.1), and let Y_4, Y_5, Y_6 denote the three ground floors of the narrow towers, i.e., towers number 1, 3 and 6 from the left in (3.1). The floors in the towers over Y_1, Y_2 and Y_3 will be called wide floors, and the other floors will be called narrow floors. Let J_i be the number of floors in the tower over Y_i. The numerical value of J_i can be read off from figure (3.1) or (3.3). The next lemma follows by inspecting (3.1) and (3.3) in conjunction with Lemma 2.1. It is an analogue of Propositions 1.2 and 1.6 in [BEK].

Lemma 3.1 *Adopt the notation and assumptions of Lemma 2.1 as well as the assumptions above. Then the following statements hold.*

$$J_k \geq 2M + 2 \textit{ for } k = 1, ..., 6. \tag{3.4}$$

$$\textit{The sets } \alpha^i(Y_k),\ i = 0,\ 1, ...,\ J_k - 1,\ k = 1, ..., 6 \textit{ form a partition of } \Omega. \tag{3.5}$$

$$\{\sigma(Y_1),\ldots,\sigma(Y_6)\} = \{\alpha^{J_1-1}(Y_1),\ldots,\alpha^{J_6-1}(Y_6)\}\,(\textit{as unordered sets}). \tag{3.6}$$

If I_1, I_2 are two floors which are adjacent in $\boldsymbol{T}$, then one is a wide floor and the other a narrow floor. (3.7)

The set $Y = Y_1 \cup Y_2 \cup \ldots \cup Y_6$ is invariant under $\alpha\sigma$, and is contained in a $\delta-$neighbourhood of $\theta/2$. (3.8)

$$\alpha\sigma(Y_k) \cap Y_k = \emptyset \textit{ for } k = 1,\ldots,6. \tag{3.9}$$

Remark 3.2
For (3.8), we assume that $\Delta\theta$ mod 1 has been chosen sufficiently small.

As for (3.6), we have $\sigma(Y_k) = \alpha^{J_k-1}(Y_k)$ for $k = 1,2,3$, and for one k in $\{4,5,6\}$, say $k = 4$, while $J_5 = J_6$ and $\alpha^{J_5-1}(Y_5) = \sigma(Y_6)$ and $\alpha^{J_6-1}(Y_6) = \sigma(Y_5)$.

Remark 3.3
We will not consider the extent to which the construction of narrow towers and Lemma 3.1 is tied up to our particular choice of partitions. Having any tower construction based on $\boldsymbol{T}_d$, the Putnam discretization of $\boldsymbol{T}$ where $\boldsymbol{T}$ is cut up along the orbit $Z\theta$, then any floor is a finite union of intervals. Hence, splitting up the towers, we may assume that all the floors are intervals. Cutting off a small, but uniform, piece around each endpoint one obtains a candidate for the floors of the narrow towers of a similar construction. However, it is not clear how one should choose the basements of the new towers in order to ensure the validity of the analogue of Lemma 3.1. As an illustration of the difficulties the reader may wish to verify that if $k < l$ and one tries to build up the narrow towers as in Figure 3.3 rather as in Figure 3.1, then the construction works if and only if $l < 0$, and even then one of the narrow towers may have smaller height than $2M + 2$.

Chapter 4

Kumjian's projections

In this section we will show that if $x_1, \ldots, x_n$ is any finite collection of elements in $C(T) \subseteq C_\theta = C(T) \times_\alpha Z \times_\sigma Z_2$ and $\varepsilon > 0$ then there exists a finite-dimensional subalgebra of C_θ which approximately contains $x_1, \ldots, x_n$ up to ε; see Lemma 4.1.

To this end, equip T with normalized Haar measure dt, and denote the unitary operators implementing α, σ on $L^2(T)$ by $u(\alpha)$, $u(\sigma)$. The C^*-algebra $C(T)$ has a faithful representation on $L^2(T)$ by pointwise multiplication, and as $C(T)$ is abelian and $Z \times_\sigma Z_2$ is amenable, C_θ is canonically isomorphic to the C^*-algebra on $L^2(T)$ generated by $C(T)$, $u(\alpha)$ and $u(\sigma)$, [Ped]. We thus identify C_θ with this algebra.

Let δ be a positive number such that

$$\delta < \triangle\theta \bmod 1 \tag{4.1}$$

Then all the floors in the new tower construction have length at least 2δ. A typical floor has the form $I = [\theta_1, \theta_2]$ where $\theta_1 = n_1\theta \bmod 1$, $\theta_2 = n_2\theta \bmod 1$ are elements in the θ-orbit. Following [Kum1], we will associate a projection p_I to I as follows:
For $t \in [\theta_i - \delta,\, \theta_i + \delta]$, put

$$\varphi_i(t) = (\theta_i + \delta - t)/2\delta, \;\; i = 1, 2, \tag{4.2}$$

and define $f_I \in C(T)$ by

$$f_I(t) = \begin{cases} 1 - \varphi_1(t) & \text{if} \qquad \theta_1 - \delta \le t \le \theta_1 + \delta, \\ 1 & \text{if} \qquad \theta_1 + \delta \le t \le \theta_2 - \delta, \\ \varphi_2(t) & \text{if} \qquad \theta_2 - \delta \le t \le \theta_2 + \delta, \\ 0 & \text{elsewhere.} \end{cases} \tag{4.3}$$

Define $g_{i,I} \in C(\boldsymbol{T})$ by

$$g_{i,I}(t) = \begin{cases} (\phi_i(t)(1-\phi_i(t)))^{1/2} & \text{if } \theta_i - \delta < t < \theta_i + 2, \\ 0 & \text{elsewhere.} \end{cases} \tag{4.4}$$

for $i = 1, 2$, and finally set

$$p_I = f_I + \varepsilon(I)(u(\alpha)^{2n_1}u(\sigma)g_{1,I} + u(\alpha)^{2n_2}u(\sigma)g_{2,I}) \tag{4.5}$$

where $\varepsilon(I) \in \{+1, -1\}$. Using that $n\theta$ is a fixed point for the homeomorphism $\alpha^{2n}\sigma$ of $\boldsymbol{T}$, one verifies that p_I is indeed a projection, whatever the sign of $\varepsilon(I)$. We now make the following choice for the sign: Put $\varepsilon(I) = +1$ if I is a wide floor, and put $\varepsilon(I) = -1$ if I is a narrow floor. This choice of sign ensures that the boundary terms of the projections belonging to adjacent intervals cancel when the projections are added up, because of (4.4), and as a consequence we have

$$\sum_I p_I = 1, \tag{4.6}$$

where the sum is over all floors in the new tower construction.

For any floor I, let t_I denote the middle point of the interval $I \subseteq \boldsymbol{T}$.

Lemma 4.1 *If $x \in C(\boldsymbol{T})$, then*

$$\lim_{N\to\infty} \|x - \sum_I x(t_I)p_I\| = 0. \tag{4.7}$$

Proof

For given $\varepsilon > 0$ choose $\delta' > 0$ such that $|t-s| < \delta' \Rightarrow |x(t) - x(s)| < \varepsilon$, and choose N, l, k etc. as in Lemma 2.1, with δ equal to this $\delta'/2$ (or choose N larger). We have

$$\begin{aligned} x &\; - \sum_I x(t_I)p_I \\ &= x - \sum_I x(t_I)f_I - \sum_I \varepsilon(I)x(t_I)(u(\alpha)^{2n_1(I)}u(\sigma)g_{1,I} + u(\alpha)^{2n_2(I)}u(\sigma)g_{2,I}). \end{aligned} \tag{4.8}$$

The functions f_I form a partition of unity on $\boldsymbol{T}$, and the support of each f_I has width at most δ'. It follows that

$$\|x - \sum_I x(t_I)f_I\| < \varepsilon. \tag{4.9}$$

As for the remaining terms, note for example that the operator $u(\alpha)^{2n_1(I)}u(\sigma)g_{1,I}$ lives on $L^2([n_1(I)\theta - \delta, n_1(I)\theta + \delta])$, and as $g_{1,I}$ is symmetric around $n_1(I)\theta$, this subspace of

$L^2(T)$ is mapped into itself by $u(\alpha)^{2n_1(I)}u(\sigma)g_1$. Also, there is a unique floor J such that $n_2(J) = n_1(I)$, i.e., the floor J that intersects I at its left endpoint. Then $\varepsilon(I)$ and $\varepsilon(J)$ have opposite sign, while

$$u(\alpha)^{2n_2(J)}u(\sigma)g_{2,J} = u(\alpha)^{2n(I)}u(\sigma)g_{1,I} \tag{4.10}$$

since $g_{2,J} = g_{1,I}$ by construction. As

$$||g_{1,I}|| \leq 1/2 \tag{4.11}$$

and

$$|x(t_I) - x(t_J)| < \varepsilon \tag{4.12}$$

it follows that

$$||x(t_I)\varepsilon(I)u(\alpha)^{2n_1(I)}u(\sigma)g_{1,I} + x(t_J)\varepsilon(J)U(\alpha)^{2n_2(J)}u(\sigma)g_{2,J}|| \leq \varepsilon/2\,. \tag{4.13}$$

Note also that the interval $[n_1(I)\theta - \delta, n_1(I)\theta + \delta]$ is disjoint from all the other intervals around the endpoints of the floors except for the floor J alluded to above. Thus the operator sum

$$\sum_I \varepsilon(I)x(t_I)(u(\alpha)^{2n_1(I)}u(\sigma)g_{1,I} + u(\alpha)^{2n_2(I)}u(\sigma)g_{2,I}) \tag{4.14}$$

decomposes into a direct sum of operators of the form

$$x(t_I)\varepsilon(I)u(\alpha)^{2n_1(I)}u(\sigma)g_{1,I} + x(t_J)\varepsilon(J)u(\alpha)^{2n_2(J)}u(\sigma)g_{2,J} \tag{4.15}$$

over all adjacent intervals J, I with J to the left. It follows from (3.16) that the norm of the operator sum is also at most $\varepsilon/2$. Combining with (4.8) and (4.9) we obtain

$$||x - \sum_I x(t_I)p_I|| < \varepsilon + \varepsilon/2 = 3\varepsilon/2 \tag{4.16}$$

and Lemma 4.1 is proved.

Chapter 5

Finite-dimensional subalgebras

We will now define a finite-dimensional subalgebra A_0 of $C(T)\times_\alpha Z\times_\sigma Z_2$ which is somewhat analogous to the A_0 of $[BEK]$, but in contrast to that case our A_o is not contained in $C(T)\times_\alpha Z$. The following lemma is analogous to Lemma 1.5 in [BEK]:

Lemma 5.1 *Let A_0 be the C^*-algebra on $L^2(T)$ generated by $p_{\alpha^i(Y_k)}$, $k=1,\ldots,6$, $i=0,\ldots,J_k-1$ and $u(\alpha)p_{T\backslash\sigma(Y)}$, where*

$$p_{T\backslash\sigma(Y)}=\sum_{k=1}^{6}\sum_{i=0}^{J_k-2}p_{\alpha_i}(Y_k)\,. \tag{5.1}$$

It follows that A_0 is finite dimensional, and the operators

$$\begin{aligned} e^k_{ij} &= u(\alpha)^i P_{y_k}u(\alpha)^{-j} \\ &= u(\alpha)^{i-j}p^j_\alpha(y_k) \end{aligned} \tag{5.2}$$

for $i,j=0,1,...,J_k-1$, $k=1,2,...,6$ constitute a complete set of matrix units for A_0. Furthermore, A_0 is invariant under $Ad(u(\sigma))$ and

$$u(\sigma)e^k_{ij}u(\sigma)^*=e^l_{J_k-1-i,\,J_k-1-j} \tag{5.3}$$

where either $k=l\in\{1,2,3,4\}$, or $\{k,l\}=\{5,6\}$.

Proof

On comparing with Lemma 3.1 and Remark 3.2, it suffices by [BEK, Lemma 1.5] to show that

$$u(\alpha)p_I u(\alpha)^*=p_{\alpha(I)} \tag{5.4}$$

and

$$u(\sigma)p_I u(\sigma)^* = p_{\sigma(I)} \tag{5.5}$$

whenever p_I, $p_{\alpha(I)}$, $p_{\sigma(I)}$ are defined from (4.5) with the same sign on ε, i.e. $\varepsilon(I) = \varepsilon(\alpha(I)) = \varepsilon(\sigma(I))$. Using the notation (4.2)–(4.5) it is clear that

$$\begin{aligned}
u(\alpha)f_I u(\alpha)^* &= f_{\alpha(I)}, \\
u(\sigma)f_I u(\sigma)^* &= f_{\sigma(I)}, \\
u(\alpha)g_{i,I}u(\alpha)^* &= g_{i,\alpha(I)}, \\
u(\sigma)g_{1,I}u(\sigma) &= g_{2,\sigma(I)}, \\
u(\sigma)g_{2,I}u(\sigma) &= g_{1,\sigma(I)}, \\
u(\alpha)u(\alpha)^{2n_1}u(\sigma)u(\alpha)^* &= u(\alpha)^{2(n_1+1)}u(\sigma), \\
u(\sigma)u(\alpha)^{2n_1}u(\sigma)u(\sigma) &= u(\alpha)^{-2n_1}u(\sigma),
\end{aligned}$$

and hence (5.4) and (5.5) follow from the definition (4.5).

Chapter 6

Homogeneous subalgebras

By adapting the techniques of [BEK] to the present circumstances, we will now prove the following:

Theorem 6.1 *Assume that θ is irrational. Given $\varepsilon > 0$ and elements $x_1, ..., x_n \in C(\mathbf{T})$, there exists a C^*-subalgebra B of $C_\theta = C(\mathbf{T}) \times_\alpha \mathbf{Z} \times_\sigma \mathbf{Z}_2$ with the same unit as C_θ such that there exist elements $y_1, ..., y_n \in B$ and a unitary $u' \in B$ with*

$$||y_i - x_i|| < \varepsilon, \quad i = 1, ..., n, \tag{6.1}$$

$$||u(\alpha) - u'|| < \varepsilon, \tag{6.2}$$

and B has the form

$$B \cong M_{J_1} \otimes C(F) \oplus M_{J_2} \oplus ... \oplus M_{J_6} \tag{6.3}$$

with $J_5 = J_6$ and J_1 even, where F is a closed subset of $\mathbf{T}$ globally invariant under complex conjugation. Furthermore, B is $Ad(u(\sigma))$-invariant, and σ acts on the canonical unitary $z \to z$ in $1_{J_1} \otimes C(F)$ by sending it into $z \to \bar{z}$. There exist matrix units e^1_{ij} for $M_{J_1} \otimes 1$ and e^k_{ij} for M_{J_k} such that

$$\sigma(e^k_{ij}) = e^k_{J_k - i - 1, J_k - j - 1} \tag{6.4}$$

for $k = 1, 2, 3, 4$, and

$$\sigma(e^k_{ij}) = e^l_{J_k - i - 1, J_k - j - 1} \tag{6.5}$$

for $\{k, l\} = \{5, 6\}$.

Before proving the Theorem we state a Corollary.

Corollary 6.2 *Assume that θ is irrational. Given $\varepsilon > 0$ and elements $x_1, ..., x_n \in C(T)$ there exists a subalgebra A of C_θ with the same unit as C_θ such that*

$$u(\sigma) \in A \tag{6.6}$$

and there exist elements $y_1, ..., y_N$ in A with

$$||y_i - x_i|| < \varepsilon \tag{6.7}$$

and a unitary $u' \in A$ with

$$||u(\alpha) - u'|| < \varepsilon, \tag{6.8}$$

and A has the form

$$A = B_0 \oplus M_{J_2} \oplus M_{J_2} \oplus M_{J_3} \oplus M_{J_3} \oplus M_{J_4} \oplus M_{J_4} \oplus M_{2J_5} \tag{6.9}$$

where

$$B_0 = \{x \in C(G, M_{2J_1});\ \ x(-1)E = Ex(-1), x(+1)E = Ex(+1)\}. \tag{6.10}$$

Here, E is a projection in M_{2J_1}, of dimension J_1, and G is a closed subset of $[-1, 1]$ (when $G \not\ni -1$ (respectively $+1$), the condition $x(-1)E = Ex(-1)$ (respectively $x(+1)E = Ex(+1)$) is vacuous.)

Proof of Corollary 6.2

As $Ad(u(\sigma))$ acts on the finite-dimensional algebra B as in Theorem 6.1, and $u(\sigma)$ is a self-adjoint unitary, it is clear that the algebra A generated by $u(\sigma)$ and B is isomorphic to a quotient of $B \times_\sigma Z_2$. Since $Ad(u(\sigma))$ restricted to the subfactors $M_{J_1} \otimes 1$, M_{J_2}, M_{J_3} and M_{J_4} leaves these factors invariant and is inner, it is clear that the corresponding components of the crossed product are $M_{J_1} \otimes 1 \oplus M_{J_1} \otimes 1$, $M_{J_2} \oplus M_{J_2}$, $M_{J_3} \oplus M_{J_3}$ and $M_{J_4} \oplus M_{J_4}$, and hence, by counting dimensions, all we have to show to prove that the corresponding components of A are isomorphic to these is that the corresponding components of $u(\sigma)$ are not contained in the matrix algebra. But it follows from (6.4) that

$$Ad(u(\sigma))|_{M_{J_k}} = Ad \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix} \equiv Ad(u_k). \tag{6.11}$$

But since σ reverses the orientation of T, it follows easily from the proof of Theorem 6.1 that if p is a minimal projection in M_{J_k} and $\sigma(p) = q$, then there are projections p_1, p_2 in C_θ such that $p_1 p_2 = 0$, $p_1 + p_2 \leq p$ and such that

$$Ad(u_k)(p_1) = \sigma(p_2), \quad Ad(u_k)(p_2) = \sigma(p_1).$$

Thus $u(\sigma)$, cut down by the central projection corresponding to M_{J_k}, is not a scalar multiple of u_k.

Next, as σ switches M_{J_5} and M_{J_6}, the algebra generated by $M_{J_5} \oplus M_{J_6}$ and the corresponding component of $u(\sigma)$ is equal to the simple crossed product M_{2J_5}. The assertion concerning B_0 is proved e.g. in [BEEK3]. The closed set G is the orbit space of T under the flip $z \longrightarrow \bar{z}$; that is, G is the projection of F into the real axis.

Proof of Theorem 6.1

The proof closely mimics the proof of Theorem 1.1 in [BEK]. First, we choose one N such that the given elements $x_1, ..., x_n$ almost lie in the algebra A_0 of Lemma 5.1. Actually, to ensure that $x_1, ..., x_n$ still are approximately contained in the modification zA_0z^* of A_0 introduced later in (6.44), we must choose N so large that $x_1, ..., x_n$ have small variation over the sets $\alpha^k(Y)$ and $\alpha^{-k}\sigma(Y)$ for $k = 0, ..., M$. Inspection of the proof of Lemma 4.1 shows that $x_1, ..., x_n$ can be approximated by linear combinations of the projections $P_{\alpha^k(Y)}, P_{\alpha^{-k}\sigma(Y)}$ for $k = 0, ..., M$ together with the P_I's corresponding to the remaining floors I. Further inspection of the proof of Lemma 4.1 shows that the approximation is uniform in the choice of Δ in (3.2) and δ in (4.1); that is, replacing δ by a smaller δ we keep the estimate, for the given N.

Now, for the moment, consider the sets

$$\begin{aligned} Y_1 &= [l\theta, \, k\theta], \\ Y_2 &= [(-k+1)\theta, \, (-N+1)\theta], \\ Y_3 &= [(-N+1)\theta, \, (-l+1)\theta], \end{aligned} \tag{6.12}$$

which are the basements in the original tower construction in Lemma 2.1. By [BEK, Lemmas 1.7 and 1.8], if Y_i is a basement such that one of the $\alpha\sigma$-fixed points $\theta/2$ or $(\theta+1)/2$ lies in the tower over Y_i, then the tower over Y_i has an even height J_i, and Y_i contains three mutually disjoint intervals A, B, C such that

$$\alpha^{J_i - 1}(A) = \sigma(A), \tag{6.13}$$

$$\alpha^{J_i - 1}(B) = \sigma(C), \tag{6.14}$$

$$\alpha^{J_i-1}(C)=\sigma(B), \tag{6.15}$$

and if k is the smallest positive integer such that $\alpha^k\sigma(A)\cap Y_i\neq\emptyset$, then

$$B=\alpha^k\sigma(A), \tag{6.16}$$

and if $0\leq j<k$ then

$$A\cap\alpha^j(A)=\emptyset. \tag{6.17}$$

Now, choose on N' so large that if $k',l'\in\{-(N'-1),...,N'-1\}$ are such that $k'\theta$ is the point in $\{-(N'-1)\theta,...,(N'-1)\theta\}$ which is closest to $N'\theta$ from above and $l'\theta$ the point which is closest to $N'\theta$ from below, then the interval $[l'\theta,k'\theta]$ is contained in the interior of A, above. Redefining A as

$$A:=[l'\theta,\ k'\theta] \tag{6.18}$$

and

$$B:=\alpha^k\sigma(A),\ \ C:=\sigma\alpha^{J_i-1}(B), \tag{6.19}$$

we see that A, B, C still has the properties (6.14)–(6.17) above; the only problem is property (6.13). To ensure this property, we must examine the proof of Lemma 1.8 in [BEK] more closely. We see that Y_1 has a $\sigma\alpha^{J_1-1}$-fixed point ω, which in our concrete setting has to be $\frac{1-J_i}{2}\theta$ or $\frac{1-J_i}{2}\theta+\frac{1}{2}$, and A is taken to be a small $\sigma\alpha^{J_i-1}$-invariant neighboorhood of ω in Y. Hence, in order that $[l'\theta,\ k'\theta]$ shall be $\sigma\alpha^{J_1-1}$-invariant, we must choose N' so that $N'\theta$ is very close to the fixed point ω. For this, let us show the following elementary lemma:

Lemma 6.3 *For $n=1,2,3,...$ let N_n be the n'th nonzero integer with the property that $N_n\theta$ is strictly closer to ω than any $k\theta$ with $|k|<|N_n|$. It follows that there exists an $n_0>0$ such that if $n\geq n_0$, then $N_n>0$ and if $k',\ l'\in\{-(N_n-1),...,(N_n-1)\}$ are such that $k'\theta$ is the point in $\{-(N_n-1)\theta,...,(N_n-1)\theta\}$ which is closest to $N_n\theta$ from above and $l'\theta$ the point which is closest to $N_n\theta$ from below, then*

$$\{k',\ l'\}=\{N_{n-1},-N_{n-1}-J_i+1\}, \tag{6.20}$$

as sets. As a consequence,

$$\sigma\alpha^{J_i-1}[l'\theta,\ k'\theta]=[l'\theta,\ k'\theta]. \tag{6.21}$$

Proof

Note that as $\sigma\alpha^{J_i-1}\omega=\omega$, the two points

$$k\theta,\ (-k-J_i+1)\theta,$$

which are conjugate under $\sigma\alpha^{J_i-1}$, have the same distance to ω. Thus, if k is an integer with $|k| > J_i - 1$ and k is negative, then $(-k - J_i + 1)$ is a positive integer with smaller absolute value than k such that $(-k - J_i + 1)\theta$ has the same distance to ω as $k\theta$. Thus, $N_n > 0$ when $|N_n| > J_i - 1$.

Let $\varepsilon > 0$ be such that if I is any interval of length ε, then the translates $\alpha^k I$, with $|k| \leq J_i$, are all disjoint. Choose n_0 so large that

$$N_{n_0-1} > J_i - 1 \tag{6.22}$$

and

$$|N_{n_0-1}\theta - \omega| < \varepsilon/4. \tag{6.23}$$

Then, if $n \geq n_0 - 1$, the translates

$$\alpha^k[N_n\theta - \varepsilon/2,\ N_n\theta + \varepsilon/2]$$

for $|k| \leq J_i$ are all disjoint, and it follows that

$$N_{n+1} > N_n + J_i\,.$$

Thus, if $n \geq n_0$, then both the points

$$N_{n-1}\theta,\ (-N_{n-1} - J_i + 1)\theta$$

lie in the set

$$\{-(N_n - 1)\theta, \dots, (N_n - 1)\theta\}$$

and also these two points are conjugate under $\sigma\alpha^{J_i-1}$. It is then clear from the definition of N_n, k', l' that

$$\{k', l'\} = \{N_{n-1}, -N_{n-1} - J_i + 1\}. \tag{6.20}$$

This ends the proof of Lemma 6.3.

By Lemma 6.3, we may redefine A as in (6.18), and still retain all the properties (6.13)–(6.17). Following [BEK, (1.33)] we now define

$$X = A \cup \alpha\sigma(A)\,. \tag{6.24}$$

The Putnam tower construction over X is then exactly like the construction over Y described in Lemma 2.1; we have just replaced N, l, k by N', l', k'. Also, as $N' > N$, the partition of $\boldsymbol{T}$ defined by the new tower construction is finer than the previous one. Now choose the Δ in (3.2) so that

$$0 < \Delta\theta \leq \frac{1}{4}\min\{(-l'-1)\theta-(-N'+1)\theta,(-N'+1)\theta-(-k'+l)\theta\} \tag{6.25}$$

and the δ in (4.1) so that

$$\delta < \Delta\theta \bmod 1, \tag{6.26}$$

with the new Δ. We will, furthermore, assume that $\Delta\theta$ and the interval A are chosen so small that when X is modified from (2.7) to (3.1), then the resulting new A is still contained in one of the three wide basements of the new Y, and the properties (6.13)–(6.17) still hold for the modified A inside the modified Y-towers. Actually, when referring to X, Y from now on, we shall refer to the modified basements in the tower construction (3.1) rather than the original basements in (2.7).

Use the new values of δ, Δ when defining A_0 from the towers over Y, and define another finite-dimensional subalgebra A_1 of C_θ by using the tower construction over X and the same values of Δ, δ. Since the partition of $\boldsymbol{T}$ defined by the towers over X is a refinement of the partition defined by the towers over Y, it is clear from definition 4.5 that the p_I's for the intervals in the Y-partition are sums of p_I's for the intervals in the X-partition, and using Lemma 5.1 it is then clear that

$$A_0 \subseteq A_1. \tag{6.27}$$

From now on, we follow [BEK, Section 1] closely, just replacing χ_I by p_I for all intervals I. So, define

$$\begin{aligned} v_0 &= \sum_{k=1}^{6}\sum_{i=0}^{J_i-2} e^k_{i+1i} + \sum_{k=1}^{6} e^k_{0J_k-1} \\ &= u(\alpha)P_{\Omega\setminus\sigma(Y)} + \sum_{k=1}^{6} u(\alpha)^{1-J_k}P_{\alpha^{J_k-1}(Y_k)} \\ &= u(\alpha)P_{\Omega\setminus\sigma(Y)} + \sum_{k=1}^{6} P_{Y_k}u(\alpha)^{1-J_k} \end{aligned} \tag{6.28}$$

and

$$\begin{aligned} u_0 &= u(\alpha)v_0^* \\ &= P_{\Omega\setminus Y} + \sum_{k=1}^{6} u(\alpha)^{J_k}P_{Y_k}. \end{aligned} \tag{6.29}$$

Then, define v_1, u_1 correspondingly from the towers over X, and verify

$$Ad(v_0u(\sigma))(v_1v_0^*) = (v_1v_0^*)^* \tag{6.30}$$

as in [BEK, Lemma 1.9]. If $X = X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5 \cup X_6$ is the partition of X defined by the new tower construction, then for any k such that σ maps the tower over X_k into itself, i.e. for $k = 1, 2, 3, 4$, the number of floors in this tower contained in Y is odd, and hence the restriction of $v_1 v_0^*$ to the corresponding central projection in A_1 has odd order, see [BEK, Lemma 1.10]. Consequently there exists a unitary operator $w \in A_1$ such that

$$wP_{\Omega\backslash Y} = P_{\Omega\backslash Y}, \tag{6.31}$$

$$w^{2M} = v_1 v_0^*, \tag{6.32}$$

$$Ad(v_0 u(\sigma))(w) = w^*, \tag{6.33}$$

$$||1 - w|| \leq \pi/2M; \tag{6.34}$$

see [BEK, Lemma 1.11], and a unitary operator $u \in A_1$ such that

$$uP_{\Omega\backslash Y} = P_{\Omega\backslash Y}, \tag{6.35}$$

$$u^M P_{Y_i} u^{-M} \geq w^{-M} P_X w^M, \tag{6.36}$$

$$Ad(v_0 u(\sigma))(u) = u, \tag{6.37}$$

$$||1 - u|| \leq \pi/M; \tag{6.38}$$

see [BEK, Lemma 1.12]. Now, defining a unitary operator z in A_1 by

$$\begin{aligned} z = &\sum_{k=0}^{M} v_0^k w^{M-k} u^{M-k} v_0^{-k} P_{\alpha^k(Y)} \\ &+ \sum_{k=0}^{M} u(\sigma) v_0^k w^{M-k} u^{M-k} v_0^{-k} u(\sigma) P_{\alpha^{-k}\sigma(Y)} \\ &+ P_{\Omega\backslash\{\bigcup_{k=0}^{M} \alpha^k(Y) \cup (\bigcup_{k=0}^{M} \alpha^{-k}\sigma(Y))\}} \end{aligned} \tag{6.39}$$

one verifies that

$$zP_{Y_i} z^* \geq P_X, \tag{6.40}$$

$$zu(\sigma) = u(\sigma) z, \tag{6.41}$$

$$zv_0z^*v_0^*P_Y = v_1v_0^*P_Y, \tag{6.42}$$

$$||zv_0z^* - v_1|| \leq 3\pi/2M; \tag{6.43}$$

see [BEK, Lemma 1.13].

Now, define

$$B = C^*(zA_0z^*, u_1), \tag{6.44}$$

where we recall that

$$u_1 = u(\alpha)v_1^*. \tag{6.45}$$

We will verify that B has the properties in Theorem 6.1. First, note that as z commutes with the projections

$$P_{\alpha^k(Y)}, \ P_{\alpha^{-k}\sigma(Y)}, \qquad k=0,\ldots,M,$$

as well as with the subprojections in A_0 of

$$P_{\Omega\setminus\{\bigcup_{k=0}^{N}\alpha^k(Y)\cup(\bigcup_{k=0}^{M}\alpha^{-k}\sigma(Y))\}},$$

it follows that all of these projections belong to zA_0z^*. Since the diameter of the set Y can be chosen arbitrarily small at the outset, it follows from Lemma 4.1 and its proof that for given $\varepsilon > 0$ and elements $x_1, ..., x_n \in C(T)$, for N large enough there exists elements $y_1, ..., y_n \in zA_0z^*$ with

$$||y_i - x_i|| < \varepsilon, \ \ i = 1, ..., n.$$

This is (6.1).

Next, as $u_1 \in B$ and $v_0 \in A_0$, we have

$$u' = u_1zv_0z^* \in B \tag{6.46}$$

and as $u(\alpha) = u_1v_1$ we have

$$||u' - u(\alpha)|| = ||zv_0z^* - v_1|| \leq 3\pi/2M \tag{6.47}$$

by (6.43). Thus, if M is chosen large enough, $u(\alpha)$ is approximately contained in B, which is (6.2). The proof of the remaining statements of Theorem 6.1 is almost identical to the end of the proof of Theorem 1.1 in [BEK]. In particular, the partial unitary

$$\begin{aligned} V &= \sum_{k=0}^{J_i-1}(zv_0^kP_{Y_i}z^*)u_1(zP_{Y_i}v_0^{-k}z^*) \\ &= \sum_{k=0}^{J_i-1}(ze_{k0}^iz^*)u_1(ze_{0k}^iz^*) \end{aligned}$$

with support $z(\sum_{k=0}^{J_i-1} e^i_{kk})z^*$ is the canonical generator of the $C(F)$-part of B in (6.3); that is, F is the spectrum of this partial unitary. As

$$ze^i_{00}z^* = zP_{Y_i}z^* \geq P_X$$

by (6.40), and u_1 acts as the identity on $P_{\Omega\setminus X}$ by (6.29), it follows that

$$F \equiv SpV = Sp(u_1).$$

In [Put] and [BEK], one now used the fact that u_1 was contained in the same K_1-class as $u(\alpha)$, which is non-trivial in $C(\boldsymbol{T}) \times_\alpha \boldsymbol{Z}$, to conclude that $F = \boldsymbol{T}$. However, in the present case the definition of the projections P_{X_k} and thus of u_1 involves the operator $u(\sigma)$, and so $u_1 \notin C(\boldsymbol{T}) \times_\alpha \boldsymbol{Z}$. Therefore we cannot conclude from this argument that $F = \boldsymbol{T}$ in our case. In the previous case one could also conclude that $Sp(u_1) = \boldsymbol{T}$ by observing that u_1 is the unitary on $L^2(X)$ which is defined by the return map on X, which is minimal as a map on the discretization of X obtained by cutting at all points on the orbit $\boldsymbol{Z}\theta$. We have not been able to turn this into an argument that the present u_1 has full spectrum.

Chapter 7

Basic building blocks

In order to prove from Corollary 6.2 that C_θ is an AF algebra, we will replace B_0 with a "large" subalgebra which is easier to describe in terms of a certain number of subalgebras which are defined as follows:

$$C_{n,k} = M_n \otimes C_k, \;\; k = -1, 0, 1, 2, \;\; n = 1, 2, \ldots, \tag{7.1}$$

where

$$C_{-1} = \boldsymbol{C}, \tag{7.2}$$

$$C_0 = C([-1, 1]) \tag{7.3}$$

= the universal C^*-algebra generated by an $x = x^*$ with $-1 \leq x \leq 1$,

$$C_1 = \{f \in C([0,1], M_2);\;\; f(0) \in \boldsymbol{C} \oplus \boldsymbol{C}\} \tag{7.4}$$

= the universal C^*-algebra generated by x, v satisfying $x = x^*$, $-1 \leq x \leq 1$, $v = v^*$, $v^2 = 1$, $vxv = -x$, and

$$C_2 = \{f \in C([-1,1], M_2);\;\; f(-1) \in \boldsymbol{C} \oplus \boldsymbol{C} \text{ and } f(1) \in \boldsymbol{C} \oplus \boldsymbol{C}\} \tag{7.5}$$

= the universal algebra generated by u, v satisfying $v = v^*$, $v^2 = 1$, $uu^* = u^*u = 1$, $vuv = u^*$.

The statements about C_1 and C_2 follow from the fact that the crossed product of $C([-1, 1])$ by the flip $(\sigma f)(x) = f(-x)$ is just C_1 and the crossed product of $C(\boldsymbol{T})$ by the flip $(\sigma f)(z) = f(\bar{z})$ is just C_2. The embedding of x, v into C_1 is given by

$$x: \;\; t \in [0,1] \to \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, \tag{7.6}$$

$$v: \ t \in [0,1] \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7.7}$$

(and then $C \oplus C$ is skewly embedded into M_2 as the two eigensubspaces of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$), and the embedding of u, v into C_2 is given by

$$u: t \in [-1,1] \rightarrow \begin{pmatrix} t + i\sqrt{1-t^2} & 0 \\ 0 & t - i\sqrt{1-t^2} \end{pmatrix}, \tag{7.8}$$

$$v: t \in [-1,1] \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.9}$$

Recall from [Kum2] that C_2 can also be characterized as the universal C*-algebra generated by two self-adjoint unitaries v_1 and v_2. The connection with the other characterization is $v = v_1, \ \ u = v_1 v_2$.

We call the following elements the canonical generators for C_k:

$$C_{-1}: \ 1 \tag{7.10}$$

$$C_0: \ x, 1 \tag{7.11}$$

$$C_1: \ x, v, 1 \tag{7.12}$$

$$C_2: \ u, v, 1 \tag{7.13}$$

Thus, $C_{n,k}$ is the universal C^*-algebra generated by elements e_{ij}, $i, j = 1, \ldots, n$ satisfying

$$e_{ij}^* = e_{ji}, \ \ e_{ij}\, e_{kl} = \delta_{jk}\, e_{il}, \ \ \sum_{i=1}^{n} e_{ii} = 1, \tag{7.14}$$

together with the canonical generators of C_k, and the latter are assumed to commute with the e_{ij}'s. We will call e_{ij}, together with the canonical generators of C_k, the canonical generators of $C_{n,k}$.

We are now ready for the reformulation of Corollary 6.2.

Corollary 7.1 *Assume that θ is irrational. Given $\varepsilon > 0$ and elements $x_1, \ldots, x_n \in C_\theta$ there exists a subalgebra A of C_θ with the same unit as C_θ such that A is a finite direct sum of basic building blocks $C_{n,k}$, and elements $y_1, \ldots, y_n \in A$ such that*

$$\|x_i - y_i\| < \varepsilon, \ \ i = 1, \ldots, m. \tag{7.15}$$

Furthermore, if one of the basic building blocks $C_{n,0}$ or $C_{n,1}$ occurs in A, then $C_{n,2}$ does not occur, and in that case there is an positive integer J such that the $C_{n,0}$'s occurring are all $C_{2J,0}$ and the $C_{n,1}$'s occurring are all $C_{J,1}$. In any case $C_{n,1}$'s occur at most twice and $C_{n,2}$'s at most once.

Proof

Referring to Corollary 6.2, it is clear that any finite subset of the algebra B_0 occurring there can be approximated by elements in a subalgebra of B_0 of the form described in the present corollary, by dividing G into sufficiently small clopen subsets.

Our next aim is to show that any separable C*-algebra with the approximation property of Corollary 7.1 is in fact an inductive limit of finite direct sums of basic building blocks.

Theorem 7.2 *Let A be a unital separable C^*-algebra, and assume that for any $\varepsilon > 0$, and any finite number $x_1, ..., x_n$ of elements in A there exists a C^*-subalgebra B of A with the same unit as A, such that B is isomorphic to a finite direct sum of basic building blocks $C_{n,k}$, and there exist elements $y_1, ..., y_n \in B$ with $||y_i - x_i|| < \varepsilon$ for $i = 1, ..., n$. Then A is an inductive limit of a sequence*

$$A_1 \to A_2 \to A_3 \to \dots$$

where each A_k is a finite direct sum of basic building blocks.

The proof of Theorem 7.2 is patterned on the proof of Theorem 2.1 in [BEK], and thus on the proofs in [Bra], [Gli]. First we establish the following lemma.

Lemma 7.3 *Let A be a unital C^*-algebra and B a C^*-subalgebra of A with the same unit as A such that B is a direct sum of basic building blocks, and let $x_1, ..., x_m \in B$.*

It follows that for any $\varepsilon > 0$ there exists a $\delta > 0$ (depending on B and $x_1, ..., x_m$) such that for any C^*-subalgebra C of A with the property that the distance of each of the generators of each of the basic building blocks of B from C is less that δ, there exists a morphism $\phi : B \to C$ with

$$||\phi(x_i) - x_i|| \leq \epsilon ||x_i|| \tag{7.16}$$

for $i = 1, ..., m$.

Proof

The proof of this lemma is almost identical to the proof of Lemma 2.3 in [BEK] or to

Lemma 4.2 in [Ell]. In either case the idea is that the relations of the generators defining B is stable in the sense that if one has a set of elements in C which approximately satisfy the relations, then they can be perturbed by a small amount to exactly satisfy the relations. We give an outline of the argument:

The first step is to approximate $x_1, \ldots, x_n$ by polynomials in the generators of the basic building blocks for B. This done, it is clear that if we have estimates like (7.16) for the canonical generators, with a smaller ε, we have the estimates (7.16) themselves. So assume that δ has been chosen small. If

$$B = \sum_{i=1}^{I}{}^{\oplus} C_{n_i, k_i} = \sum_{i=1}^{I}{}^{\oplus} M_{n_i} \otimes C_{k_i},$$

where the sum is finite, consider the finite dimensional subalgebra

$$B_0 = \sum_{i=1}^{I}{}^{\oplus} M_{n_i} \otimes 1,$$

and let $e_{jl}^{n_i}$ be a complete set of matrix units for B_0. By [Gli, Lemma 1.10] or [Bra, Lemma 2.1] there exists a set of matrix units $f_{jl}^{n_i}$ in C such that $e_{jl}^{n_i}$ is close to $f_{jl}^{n_i}$ for each n_i, j, l, and these matrix units span a subalgebra C_0 of C which is isomorphic to B_0. By integrating $Ad(u)$ over u in the unitary group of C_0, it is clear that we can approximate the x, u, v-generators by elements in the relative commutant $C_0' \cap C$ of C_0 in C, and by cutting these down by the central projections $f^{n_i} = \sum_j f_{jj}^{n_i}$ in C_0, we may also assume that the approximants sit inside the appropriate central projection. Hence, by universality of the algebras C_0, C_1, C_2 the problem of defining ϕ boils down to showing that if the relations defining these algebras are approximately verified by some elements, a small perturbation of these elements will exactly verify the relations. For C_0 this is trivial, for C_2 the argument is essentially given in the proof of Lemma 2.3 in [BEK], so let's do C_1: Assume that we have the approximate relations

$$x \cong x^*, \ \ ||x|| \overset{<}{\cong} 1, v \cong v^*, \, v^2 \cong 1 \text{ and } vxv \cong -x.$$

First take the self-adjoint part of v and modify it by spectral theory so that $v = v^*$ and $v^2 = 1$. Then take the self-adjoint part of x and modify x by spectral theory so that $x = x^*$ and $||x|| \leq 1$. Then, as the new v, x are close to the old ones, $vxv \cong -x$ even after modification. Hence the element $\frac{1}{2}(x - vxv)$ is close to x, and replacing x by this latter element we exactly obtain $vxv = -x$.

This ends the proof of Lemma 7.3.

Proof of Theorem 7.2

The proof of Theorem 7.2 from Lemma 7.3 is now almost a word-for-word rendering of the proof of Theorem 2.1 in [BEK] from Lemma 2.3 there, with the difference that the morphisms in the inductive system are no longer necessarily injective. Apart from Lemma 7.3, the only input in the proof is separability. A similar proof is the proof of Theorem 4.3 from Lemma 4.2 in [Ell].

Corollary 7.4 *Assume that θ is irrational. Then the algebra C_θ is the inductive limit of a sequence of algebras which are finite direct sums of basic building blocks $C_{n,k}$. Furthermore, there are the same restrictions on the basic building blocks actually occurring in one of the algebras in the sequence as in the concluding remarks of Corollary 7.1.*

Proof

This is clear from Corollary 7.1 and Theorem 7.2, and the proof of Theorem 7.2.

Chapter 8

Small eigenvalue variation

In this section we will prove Theorem 1.1 by combining techniques from [BBEK] and [Su]. Actually, Theorem 1.1 follows from the following theorem in conjunction with Corollary 7.4.

Theorem 8.1 *Let C be a simple unital C^*-algebra with a unique trace state, and assume that C is the inductive limit of a sequence of algebras which are finite direct sums of basic building blocks $C_{n,k}$. It follows that C is an AF-algebra.*

Proof
Our basic building blocks are a subclass of the basic building blocks considered in [Su], which are C*-subalgebras of $C(\Omega, M_n)$ where Ω is a finite connected graph such that the subalgebra has diagonal block form at some vertices in Ω. It is proved in [Su], Theorem 1 that if C has real rank zero, then $K_*(C)$ with the graded dimension range is a complete invariant for C. For our special basic building blocks, $K_1 = 0$, and hence it follows from Su's classification that our algebras are AF if they have real rank zero. To prove that C has real rank zero, we just copy the proof of $1 \Rightarrow 5$ in Theorem 1.3 of [BBEK], where the same thing is proved in the case that the basic building blocks are full homogeneous algebras over spaces of dimension at most 2; that is, one first establishes small eigenvalue variation and then proves that C has real rank zero. We omit the details, but would also like to remark that one could prove directly that C is an AF-algebra from small eigenvalue variation by essentially the same argument as in [BEK].

This argument also occurs in [Ell 2].

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